



Hamiltonian analysis of an on-shell $U(1)$ gauge field theory



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ABSTRACT

We perform the Hamiltonian analysis of an on-shell $U(1)$ gauge field theory, in which the action is not invariant under local $U(1)$ transformations but recovers the invariance when the equations of motion are imposed. We firstly apply Dirac's method of Hamiltonian analysis. We find one first-class constraint and two second-class constraints in the vector sector. It implies the photons have only two polarisations, at least at the classical level, although the standard $U(1)$ symmetry is explicitly broken. The reduced Hamiltonian is bounded from below and the on-shell $U(1)$ gauge field theory is free from ghosts at the classical level.

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1. Introduction

In quantum field theory, the local $U(1)$ gauge symmetry is the most established symmetry underlying the interaction between matter and the electromagnetic field. There are several profound physical implications including the current conservation and the absence of longitudinal mode of photon and so on. At the quantum level, the Ward–Takahashi identity protects the masslessness of photons against quantum loop corrections and thus a photon can find no rest.

At high energy scales, the break-down of perturbation theory due to the Landau pole singularity implies that our QED is just a low energy effective theory and it should be integrated into a larger symmetry group of $SU(2) \otimes U(1)$ [1]. On the other hand, the theory exhibits less symmetry at ultra low energy scale. For example, as known in condensed matter physics, pairs of electrons may be driven into boson condensates described by a $U(1)$ -charged complex scalar field at low energy scales. Below a critical temperature, the scalar field develops a non-trivial VEV and the $U(1)$ symmetry is spontaneously broken. As a consequence, some materials exhibit superconductivity [2]. A theory with an explicitly broken $U(1)$ symmetry was proposed by Proca in his work on the massive spin-1 boson field [3]. In addition to the 2 transverse degrees of freedom, the massive spin-1 particle has the longitudinal mode, which decouples from the scattering process in the massless limit.

It is then intriguing to ask whether there are some other symmetry breaking mechanisms of which the broken phase is less symmetric than QED but more symmetric than the Proca theory? Inspired by this question, we come up with an idea of a “weakly broken” gauge symmetry. By “weakly broken” we mean that the action of our theory is invariant under local $U(1)$ transformations only when the equations of motion are imposed. We call this theory an on-shell $U(1)$ gauge field theory. A very natural and direct question is that how many degrees of freedom are there in this type of gauge field theory? This question should be addressed at both classical and quantum levels. In this paper, we consider an interesting realisation of such a theory and perform the Hamiltonian analysis at classical level. The computation of quantum corrections is deferred to future work.

This paper is organised as follows: In Sec. 2, we will briefly introduce the idea of on-shell gauge symmetry. In Sec. 3, we apply Dirac's method of Hamiltonian analysis and show that the number of degrees of freedom remains the same as the usual scalar plus $U(1)$ gauge theory. We derive the reduced Hamiltonian in Sec. 4, and show that the Hamiltonian is bounded from below. Conclusions and further discussions are given in Sec. 5.

2. The on-shell $U(1)$ gauge symmetry

In our theory, there is a scalar field which breaks the local $U(1)$ symmetry explicitly but “weakly”. The Lagrangian density is written as

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$$\mathcal{L} = -\frac{1}{4}g^2(\phi)F_{\mu\nu}F^{\mu\nu} - e_0f(\phi)A_\mu\bar{\psi}\gamma^\mu\psi + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \quad (1)$$

where $g(\phi)$ and $f(\phi)$ are two generic and dimensionless function of a scalar field ϕ . For simplicity, we only consider a single charged Dirac fermion, say electron, as a representative of matter, and ignore the gravitational degrees of freedom since they are irrelevant to the current issue. The local $U(1)$ symmetry of QED is broken due to the abnormal gauge coupling $e_0f(\phi)A_\mu\bar{\psi}\gamma^\mu\psi$.

This model, with the special choice $f(\phi) = g(\phi)$, was initially proposed to explain the ubiquitous presence of cosmic magnetic fields [4]. The pre-factor $g(\phi)^2$ in front of $F_{\mu\nu}F^{\mu\nu}$ increase exponentially and compensate the redshift factor of the magnetic field due to the expansion during inflation, while the strong coupling between the gauge field and the fermion can be avoided by setting $f(\phi) = g(\phi)$ in $e_0f(\phi)A_\mu\bar{\psi}\gamma^\mu\psi$. Related to our model, a similar $U(1)$ symmetry breaking model was discussed earlier by Dvali [5], which was motivated to solve the gauge hierarchy problem, i.e. why the electroweak scale is so much lower than the Planck scale. In the model of Ref. [5], a three-form field interacts with 2-branes, where the charge of the latter is a function of a scalar field and thus the local gauge symmetry is also explicitly broken.

To understand the internal symmetry of this theory, let us firstly derive the equations of motion of the vector field,

$$\partial_\mu (g^2 F^{\mu\nu}) - e_0f\bar{\psi}\gamma^\nu\psi = 0. \quad (2)$$

Taking the divergence of the above, by noting the identity $\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0$, we must have

$$\partial_\mu (e_0f\bar{\psi}\gamma^\mu\psi) = 0. \quad (3)$$

On the other hand, the global $U(1)$ symmetry in the fermion sector implies the existence of a conserved Noether current,

$$\partial_\mu (\bar{\psi}\gamma^\mu\psi) = 0. \quad (4)$$

Combining eqs. (3) and (4), we obtain the following “constraint” equation:

$$\bar{\psi}\gamma^\mu\psi\partial_\mu f = 0. \quad (5)$$

Under the local $U(1)$ gauge transformation,

$$A_\mu \rightarrow A_\mu + \partial_\mu\chi, \quad \psi \rightarrow e^{-ie_0f\chi}\psi, \quad (6)$$

the variation of the action reads

$$\delta_\chi S = e_0 \int d^4x \chi (\bar{\psi}\gamma^\mu\psi\partial_\mu f) + \text{total derivatives}. \quad (7)$$

Where δ_χ denotes the variation $\delta_\chi A_\mu = \partial_\mu\chi$. With Eq. (5) imposed, the variation of the action vanishes. Therefore, our action is invariant under local $U(1)$ gauge transformations only when the equations of motion are imposed.

Here two comments are in order. One may be confused by our statement about the on-shell gauge symmetry. One may think that with help of equation of motion, any actions are invariant under the transformation of fields. However, that is only the case when we perform the linear and infinitesimal transformation of the field. As a gauge invariant theory, the action must be invariant at fully non-linear level. In our theory, we do need equations of motion to prove that our theory is invariant at fully non-linear level. On the other hand, Since it looks like eq. (5) constraints the system too much, one might worry if it would render the dynamics trivial. In the following, we show this is not the case. We show that the system has just the necessary and sufficient number of dynamical degrees of freedom. In fact, since eq. (5) guarantees the on-shell $U(1)$ symmetry as mentioned above, it could be regarded as a constraint that kills the longitudinal component of the vector field which would be present if $U(1)$ were totally broken.

3. Dirac's method of Hamiltonian analysis

In this section, we apply Dirac's method of Hamiltonian analysis [6] to our theory (1). The conjugate momenta read

$$\begin{aligned} \pi_\phi &= \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = \dot{\phi}, & \pi_i &= \frac{\delta\mathcal{L}}{\delta\dot{A}_i} = g^2(\dot{A}_i - \partial_i A_0), & \pi_0 &= \frac{\delta\mathcal{L}}{\delta\dot{A}_0} = 0, \\ \pi_\psi &= \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^\dagger, & \pi_{\psi^\dagger} &= \frac{\delta\mathcal{L}}{\delta\dot{\psi}^\dagger} = 0. \end{aligned} \quad (8)$$

The Hamiltonian density is thus given by

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}\pi_\phi^2 + \frac{1}{2}\partial_i\phi\partial_i\phi + V(\phi) + \frac{1}{2}\pi_i\pi_i + g^2(\partial_i A_j - \partial_j A_i)^2 - A_0\partial_i\pi_i \\ &\quad + e_0f(\phi)A_\mu\bar{\psi}\gamma^\mu\psi - i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi + \lambda_0\pi_0 + \lambda_\psi(\pi_\psi - i\psi^\dagger) + \pi_{\psi^\dagger}\lambda_{\psi^\dagger}, \end{aligned} \quad (9)$$

with the 3 primary constraints,

$$\Phi_1 = \pi_0 \approx 0, \quad \Phi_2 = \pi_\psi - i\psi^\dagger \approx 0, \quad \Phi_3 = \pi_{\psi^\dagger} \approx 0, \quad (10)$$

where ≈ 0 denotes the equality when the constraints are satisfied. To be consistent, we demand the Poisson brackets of these 3 primary constraints with the Hamiltonian to vanish, i.e. $\{\Phi_i, H\}_{P.B.} \approx 0$.

Let us first look at the fermion sector,

$$\{\Phi_2, H_0\}_{P.B.} = -i\lambda_{\psi^\dagger} - \left[e_0 g(\phi) A_\mu \bar{\psi} \gamma^\mu + i\partial_i \left(\bar{\psi} \gamma^i \right) + m\bar{\psi} \right], \quad (11)$$

$$\{\Phi_3, H_0\}_{P.B.} = i\lambda_\psi - \left[e_0 g(\phi) A_\mu \gamma^0 \gamma^\mu \psi - i\gamma^0 \gamma^i \partial_i \psi + m\gamma^0 \psi \right]. \quad (12)$$

These 2 Poisson brackets fix the coefficients λ_ψ and λ_{ψ^\dagger} and therefore they do not give any new constraints. Note that these 2 primary constraints, as well as the determined values of λ_ψ and λ_{ψ^\dagger} are essential to obtain the correct equations of motion for the fermion.

Now let us compute the Poisson brackets of the first primary constraint with the Hamiltonian. For the primary constraint $\pi_0 \approx 0$, we have consistency condition,

$$\{\Phi_1, H_0\}_{P.B.} = e_0 f(\phi) \psi^\dagger \psi - \partial_i \pi_i \equiv \Phi_4 \approx 0. \quad (13)$$

Thus the consistency condition gives us a secondary constraint and we name it Φ_4 . Then we plug it into the Hamiltonian and treat it as the same footing as the primary constraints,

$$\mathcal{H}_1 \equiv \mathcal{H}_0 + \lambda_4 \Phi_4. \quad (14)$$

The consistency condition requires that the Poisson bracket of this new secondary constraint Φ_4 with the Hamiltonian must also vanish, and it further gives one more secondary constraint,

$$\{\Phi_4, H_1\}_{P.B.} = e_0 f' \left[\psi^\dagger \psi \cdot \pi_\phi + \bar{\psi} \gamma^i \psi \partial_i \phi \right] \equiv \Phi_5 \approx 0. \quad (15)$$

Again, we plug the new secondary constraint into the Hamiltonian, and treat it as the same footing as the primary constraints. Then we obtain the total Hamiltonian

$$\mathcal{H}_T \equiv \mathcal{H}_1 + \lambda_5 \Phi_5. \quad (16)$$

The consistency condition

$$\{\Phi_i, H_T\}_{P.B.} \approx 0, \quad \text{where} \quad i = 1, 2, 3, 4, 5, \quad (17)$$

fixes the coefficients λ_ψ , λ_{ψ^\dagger} , λ_4 and λ_5 and no new secondary constraints are generated.¹ We also find that

$$\{\Phi_1, \Phi_i\} \approx 0, \quad \text{where} \quad i = 1, 2, 3, 4, 5. \quad (18)$$

Therefore, $\Phi_1 \approx 0$ is the first-class constraint. On the other hand,

$$\{\Phi_m, \Phi_n\} \simeq \begin{pmatrix} 0 & -i & e_0 f \psi^\dagger & -(e_0 f' \psi^\dagger \pi_\phi + e_0 \bar{\psi} \gamma^i \partial_i f) \\ i & 0 & e_0 f \psi & -(e_0 f \psi \pi_\phi + e_0 \gamma^0 \gamma^i \psi \partial_i f) \\ -e_0 f \psi^\dagger & -e_0 f \psi & 0 & -(e_0 f' \psi^\dagger \psi)^2 \\ (e_0 f' \psi^\dagger \pi_\phi + e_0 \bar{\psi} \gamma^i \partial_i f) & (e_0 f \psi \pi_\phi + e_0 \gamma^0 \gamma^i \psi \partial_i f) & (e_0 f' \psi^\dagger \psi)^2 & 0 \end{pmatrix},$$

where $m, n = 2, 3, 4, 5$. (19)

The determinant of the above matrix is non-vanishing,

$$\det\{\Phi_m, \Phi_n\} = -(e_0 f' \psi^\dagger \psi)^4 \neq 0, \quad (20)$$

and therefore all of them are second-class.

Now we are ready to count the number of the degrees of freedom. Firstly, the number of degrees of fermion is the same as the one in standard QED, $\Phi_2 \approx 0$ and $\Phi_3 \approx 0$ are second class, they fix the 1-to-1 correspondence between canonical variables and their conjugate momenta. Therefore the fermion has 4 states, 2 for electron and another 2 for positron, which is the same as the standard QED. In the vector field sector, the Hamiltonian contains 8 canonical conjugate pairs. We have 1 first-class constraint and 2 second-class constraints. Together they kill another 4 degrees in phase space, thus the number of degrees of freedom in the phase space of the electromagnetic field is 4. The corresponding number of physical degrees of freedom is thus 2. Adding the canonical conjugate pair of the scalar ϕ , we conclude that the number of degrees of freedom in our theory is the same as the usual scalar plus $U(1)$ gauge theory. Namely, no longitudinal mode is found in the photon of our theory.

The Hamiltonian analysis has helped us to understand the physical meaning of on-shell $U(1)$ symmetry. The on-shell $U(1)$ symmetry gives us two second class constraint equations (or in other words, conservation laws), they eliminate the longitudinal mode of photon.

¹ We need to be careful here $\{\Phi_5, H\}_{P.B.} \supset \{\Phi_5, \int \lambda_5 \Phi_5 d^3x\} + \dots = -\lambda_5 e_0 g' \bar{\psi} \gamma^i \psi \partial_i (e_0 g' \psi^\dagger \psi) + \dots$ and thus the consistency condition of $\Phi_5 \approx 0$ fixes λ_5 .

4. Reduced Hamiltonian and ghost freeness

We have proved that our theory has correct number of degrees of freedom in the previous sections. The validity of the theory, at least at classical level, requires the Hamiltonian must be bounded from below. Since we are dealing with system with constraints, we have to integrate out all constraints, and derive the reduced Hamiltonian. In this approach, one rewrites the Lagrangian in the first order form, and reduces it by plugging all the solutions of the constraint equations into it, irrespective of whether they are first-class or second-class.

Again, we start from the Lagrangian (1). The conjugate momenta read

$$\pi_\phi = \dot{\phi}, \quad \pi_i = g^2(\phi)(\dot{A}_i - \partial_i A_0), \quad \pi_0 = 0, \quad \pi_\psi = i\psi^\dagger, \quad \pi_{\psi^\dagger} = 0. \quad (21)$$

As before the last 3 momenta form 3 primary constrains. Plugging them into the Lagrangian, the Hamiltonian reads

$$\begin{aligned} H &= \frac{1}{2}\pi_\phi\pi_\phi + \frac{1}{2}\partial_i\phi\partial_i\phi + V(\phi) + \frac{1}{2g^2}\pi_i^2 + \frac{1}{2}g^2\partial_i A_j\partial_i A_j - \frac{1}{2}g^2\partial_i A_j\partial_j A_i + e_0 f(\phi)A_i\bar{\psi}\gamma^i\psi \\ &\quad - i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi + A_0\left[-\partial_i\pi_i + e_0 f(\phi)\psi^\dagger\psi\right] \\ &\equiv H_1 + A_0\left[-\partial_i\pi_i + e_0 f(\phi)\psi^\dagger\psi\right], \end{aligned} \quad (22)$$

where and in what follows, $\bar{\psi} = \psi^\dagger\gamma^0 = -i\pi_\psi\gamma^0$ is understood. The Lagrangian in the first order form is given by

$$\mathcal{L} = \pi_\phi\dot{\phi} + \pi_i\dot{A}_i + \pi_\psi\dot{\psi} - H_1 + A_0\left[\partial_i\pi_i - e_0 f(\phi)\psi^\dagger\psi\right]. \quad (23)$$

As clear from the above, there appears a secondary constraint with A_0 as a Lagrange multiplier,

$$\partial_i\pi_i = e_0 f(\phi)\psi^\dagger\psi. \quad (24)$$

To solve this constraint we decompose the gauge field and its conjugate momentum into the transverse and longitudinal parts,

$$A_i = A_i^T + \partial_i\chi, \quad \pi_i = \pi_i^T + \partial_i\pi, \quad (25)$$

where $\partial^i A_i^T = \partial^i \pi_i^T = 0$. Then the constraint (24) can be rewritten as

$$\pi = \Delta^{-2}\left(e_0 f\psi^\dagger\psi\right), \quad (26)$$

where $\Delta = \partial^i\partial_i$ and Δ^{-2} is a non-local operator defined in the way such that $\Delta\Delta^{-2}Q = Q$ with an appropriate boundary condition. The Lagrangian is now rewritten as

$$\mathcal{L}_1 = \pi_\phi\dot{\phi} + \pi_i^T\dot{A}_i^T + \pi_\psi\dot{\psi} - H_{red} + \chi e_0\left[-\partial_t\left(f\psi^\dagger\psi\right) + \partial_i\left(f\bar{\psi}\gamma^i\psi\right)\right], \quad (27)$$

where we have employed integration by parts appropriately and used Eq. (26). The H_{red} is the reduced Hamiltonian,

$$\begin{aligned} H_{red} &= \frac{1}{2g^2}\pi_i^T\pi_i^T + \frac{1}{g^2}\pi_i^T\partial_i\pi + \frac{1}{2g^2}\partial_i\pi\partial_i\pi \\ &\quad + \frac{1}{2}g^2\partial_i A_j^T\left(\partial_i A_j^T - \partial_j A_i^T\right) - i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi + e_0 f A_i^T J^i \\ &\quad + \frac{1}{2}\pi_\phi\pi_\phi + \frac{1}{2}\partial_i\phi\partial_i\phi + V(\phi), \end{aligned} \quad (28)$$

where $J^i \equiv \psi^\dagger\gamma^i\psi$. In the Lagrangian eq. (27), we have spotted another constraint with χ as Lagrangian multiplier. Note that this term is nothing but the generalised current conservation equation (3) that reflects the (on-shell) $U(1)$ gauge symmetry. Therefore, we can simply remove this constraint from the Lagrangian and the reduced Hamiltonian gives us all correct equations of motion. One can check that the equations of motion given by the reduced Hamiltonian H_{red} are the same as the ones given by the original Lagrangian eq. (1). For instance, in the gauge field sector, we have

$$\dot{A}_i^T = \frac{\partial H_{red}}{\partial \pi_i^T} = \frac{1}{g^2}\pi_i^T + \frac{1}{g^2}\partial_i\pi, \quad (29)$$

$$\dot{\pi}_i^T = -\frac{\partial H_{red}}{\partial A_i^T} = \partial_j\left(g^2\partial_i A_j^T - g^2\partial_j A_i^T\right) - e_0 f J^i. \quad (30)$$

These equations of motion are the same as the ones from the original Lagrangian after adopting the Coulomb gauge $A_0 = 0$ and $\chi = 0$. Similarly, we can also check that the equation of motion of scalar field ϕ and fermions are also consistent with the ones derived from the original Lagrangian. Note that the reduced Hamiltonian H_{red} is in the quadratic form and thus bounded from below. We thus conclude that our theory is ghost free at the classical level.

5. Conclusion and discussion

In this paper, we considered a novel type of $U(1)$ gauge field theory, of which the action is invariant under local $U(1)$ gauge transformations only when the equations of motion are imposed. We call it an on-shell $U(1)$ gauge theory. We applied Dirac's method of Hamiltonian analysis to clarify the number of degrees of freedom in this theory. We have spotted 1 first-class constraint and 2 second-class constraints in the gauge field sector, in contrast to the stand QED which has 2 first class constraints. Apart from this subtle difference, the result is that the number of degrees of freedom in the photon sector remains the same, that is, there are only 2 transverse degrees of freedom despite the fact that the local $U(1)$ symmetry is broken. Adding the degrees of freedom of the fermion and scalar sectors, we conclude that the number of degrees of freedom in our theory is the same as the usual scalar plus $U(1)$ gauge theory. No longitudinal mode is found in the photon of our theory. Nor the seemingly new constraint (5) does not kill the physical degrees of freedom. We have also derived the reduced Hamiltonian in the Coulomb gauge and found that it is bounded from below. Therefore our theory is free from the ghost problem.

We should mention that our analysis is classical, and thus it is still premature to claim the validity of our on-shell $U(1)$ gauge theory. It is necessary to check whether a quantum anomaly appears at loop level and spoils our gauge symmetry. And if so, if there is a mechanism or a new ingredient to cancel the anomaly. We plan to come back to this issue in our future work.

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